AN INFERENTIAL VIEW ON CONCEPT FORMATION

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This paper focuses on an inferential view on introducing new concepts in mathematics classrooms. A theoretical framework is presented which helps to analyse and reflect on the processes of teaching and learning mathematical concepts. The framework is based on the philosophies by Ludwig Wittgenstein and Robert Brandom. Wittgenstein's language-game metaphor and especially its core, the primacy of the use of words, provide insight into the processes of giving meaning to words. Concerning the inferentialism by Brandom, the use of words in inferences can be regarded as an indicator of the understanding of a concept. The theoretical considerations are exemplified by the interpretation of a scene of real classroom communication.

INTRODUCTION

A lot of research on communication in the mathematics classroom has been done. Mathematical interactions have been analysed from many different perspectives (cf. Cazden, 1986). This paper focuses on the teaching and learning of mathematical concepts in classroom communication. By his theory of "language-games", Wittgenstein offers an alternative view on the introduction of concepts in mathematics classrooms. Elements of his perspective have often been used to discuss problems concerning communication in the mathematics classroom (e.g., Bauersfeld, 1995; Schmidt, 1998; Sfard, 2008). According to Wittgenstein, the expression of words does not constitute their meaning. Rather, it is the use of words, which constitutes the meaning, and therefore, the use of words constitutes the concept. On the basis of Wittgenstein's philosophy, the American philosopher Brandom worked out an inferential approach to the comprehension of the processes of concept formation.

On the basis of the theory of analysing arguments by Toulmin it will be described in this article in how far the processes of concept formation can be analysed in accordance with an understanding like this. In particular, the significance of judgments (the combination of subjects and predicates) and their connection among one another during their concept formation will be focused.

USE OF WORDS IN LANGUAGE-GAMES

Wittgenstein's concept of language-game is closely connected with the process of concept formation. It means that words do not have a meaning by themself. Therefore, a fixed, temporal lasting word's meaning does not exist:

"Naming is so far not a move in the language-game—any more than putting a piece in its place on the board is a move in chess. We may say: *nothing* has so far been done,

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when a thing has been named. It has not even got a name except in the language-game.” (Wittgenstein, PI § 49)

Thus, Wittgenstein has a complex opinion on processes of concept formation. That means that the meaning of a word is solely put down to its use: “For a *large* class of cases—though not for all—in which we employ the word ‘meaning’ it can be defined thus: the meaning of a word is its use in the language” (PI § 43).

The meaning of a word shows and manifests itself in using the word in language. This might be a reason for the fact that Wittgenstein does not define what exactly he understands by speaking of “language-games”. He uses the word “language-game” by describing the use of this word (e.g., by giving examples). That way, he gives meaning to this word. The theory of Wittgenstein of the attribution of meaning through the use of words is also closely connected with those of the language-game in another way: To this, let us have a look on the concept of numbers: When students understand numbers as a quantitative aspect of objects, then they can use this for calculating. But the handling of numerals is changing when numbers are regarded as ordinal numbers. Now, operations cannot be used in such an easy way anymore. The comprehension of the cardinal aspect of numbers is not sufficient either when negative numbers are introduced. Each of these changes entails an alteration of the language-game. In the changing language-games, the same numbers can be used in different ways. The way of use determines the current meaning. However, a well-developed concept of numbers needs different kinds of comprehensions – that is different ways of use – which are connected with *family resemblances*, to say it in Wittgenstein’s words:

> And for instance the kinds of number form a family in the same way. Why do we call something a ‘number’? Well, perhaps because it has a—direct—relationship with several things that have hitherto been called number; and this can be said to give it an indirect relationship to other things we call the same name. And we extend our concept of number as in spinning a thread we twist fibre on fibre. And the strength of the thread does not reside in the fact that some one fibre runs through its whole length, but in the overlapping of many fibres. (Wittgenstein, PI § 67)

The use of words in a language-game is by no means arbitrary. Rather, the use is determined by certain rules. These rules tell us how words can be applied:

> We can say that a language is a certain amount of activities (or habits) which are determined by certain rules, namely those rules that rule all the different ways of use of words in language. (Fann 1971, p. 74; my own translation)

Accordingly, observing the rules, that determine the use of words, is a considerable feature of our linguistic acting. A rule has the function of a “sign-post” (Wittgenstein, PI § 85) although each rule can be interpreted in a different way. Within the mathematics education research, a lot of rules, which determine the language-game “*mathematics education*”, have already been reconstructed. The patterns of interaction and routines which were described by Voigt (1984) can also be counted as (combinations of) rules. For instance, *the pattern of staged-managed everyday*

occurrences (in German: "*Muster der inszenierten Alltäglichkeit*") describes the *as if*-character of classroom situations in which the students' extracurricular experiences are taken up: if the students make too much use of these experiences, the teacher is going to disregard this use and highlights the mathematical contents. By using the word *demathematizing* (in German: "*Vermathematisierung*"), Neth and Voigt (1991) describe how teacher – while working on an situation which is open for different kinds of interpretations – makes a note on single words, formulas, signs, or the like of the students, in order to funnel the students' diversity of interpretation on mathematics as quick and purposeful as possible. Such rules make sure that the actions in class run smoothly by showing the agents, for instance, which actions they have to carry out, what they can achieve with them and where the limits of their actions are. Therefore, rules are constitutive for the classes, particularly as they determine the use of words or rather sentences on the one hand and support that the classes pass off smoothly on the other hand.

INFERENTIAL USE OF WORDS

Following Wittgenstein, a concept can be developed, if different ways of using the relevant word are well known. The definition of a word is just one possible way of using this word. Knowing different ways of using a word includes, among other things, knowing and using sentences that go with them:

> 'Owning' a mathematical term requires to know more relations and to know more about the handling with the term than it is expressed in its definition. […] Proofs help to explain the terms' inner structures as well as to link concept and with that to develop the purport of term. (Fischer & Malle 1985, S. 189f, my own translation)

We use words in situations of giving reasons for statements – also statements in which this word is used. For example, we can use "commutative law" to give reason for the similarity of 9+4 and 4+9. The aspect of reason of concept formation shows itself in the structure of the potential words' ways of use. Thus, every definition, for instance, has a conditional structure ("If…, then…"). Definitions are equivalence relations (or rather biconditional – "if and only if") which are also used in arguments. In short: The words' meanings are arranged in an inferential way. The American philosopher Brandom elaborated this inferential approach: "To talk about concepts is to talk about roles in reasoning." (ibid. 2000, p. 11). The understanding of a word is described by Brandom as follows:

> Grasping the concept that is applied in such a making explicit is mastering its *inferential* use: knowing (in the practical sense of being able to distinguish, a kind of knowing *how*) what else one would be committing oneself to by applying the concept, what would entitle one to do so, and what would preclude such entitlement. (ibid.)

The inferential use is carried out using reasoned arguments in situations of reason. To examine the students' corresponding arguments, the Toulmin-scheme – which has been already become established in mathematical education research – can be used. It also helps to reconstruct the implicit shares of arguments. In accordance with this, an

argument consists of several functional elements. Undisputable statements function as *datum* (Toulmin 1996, p. 88). Coming from this, a *conclusion* (ibid.) can be inferred, which might have been a doubtful statement before. The *rule*[1] shows the connection between datum and conclusion. The rule legitimizes the conclusion. If the rule's validity is questioned, then the arguer could be forced to assure it. Within the reconstruction, such making safes are recorded as *backings* (ibid, 93ff) and can happen, for instance, in giving further details about the field where the rule comes from. As an example for the analysis by means of the argumentation-scheme by Toulmin, the following fictitious remark of a student is reconstructed, which functions at the same time as an example for the inferential use of the concept *bigger*: "As 3 apples are more than 2 apples, 3 is bigger than 2." According to this statement that - talking about numbers of apples – there is a smaller-bigger relation (datum), it can be concluded that there is a relation of size between the relevant numbers (conclusion). The conclusion is legitimized by a rule which is only implicit and which can be supported with the reference to the aspect of cardinal numbers (backing). Accordingly, the following Toulmin-scheme can be reconstructed:

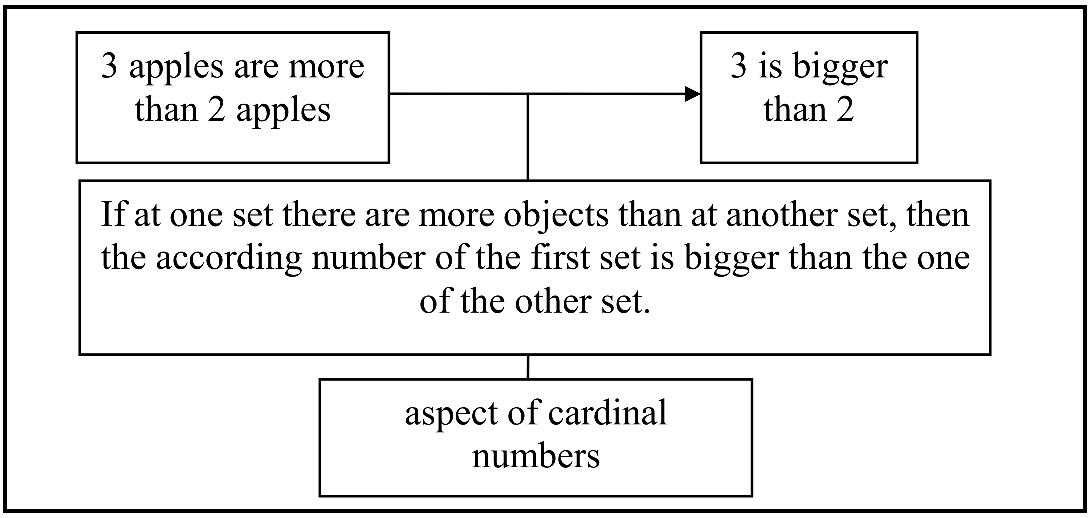

Figure 1: Application of the Toulmin-scheme

Following Wittgenstein, by means of such an argument a relation between two concrete numbers is expressed. In certain language-games, such an argument is surely regarded to be valid. But introducing negative numbers at school means that such kind of use of the word *bigger* is possibly no longer accepted. This change of the language-game causes a different use of numbers. Although 3 apples are more than 2 apples is true, it does not mean that -3 is bigger than -2. If the rule is applied on negative numbers in this way, it loses its' validity.

With regard to the theoretical consideration before, different important elements of the processes of concept formation can be recognized:

[1] The general connection which is described as warrant by Toulmin corresponds not entirely to the above rules by Wittgenstein (cf. the examples of (combinations of) rules given above).

- *datum* and *conclusion* consist of judgments, as a link between subjects and predicates,
- *rules* have a general character, in so far they connect general judgments in conditional or biconditional forms and
- the *backings* which are the basis for an argument.

Accordingly, the enormous significance of concrete and (combinations of) general judgments for concept formation is shown: Concrete judgments (datum and conclusion) are linked via more general connections (rules). The possibility of this connection is based on the knowledge of an area or rather of a context in which this connection is perceptible to the learners (backing).

METHODOLOGY

According to Wittgenstein we should not ask: What is the meaning of a word? Rather, we should analyse what kind of meaning a word gets (by its use) in the classroom. Therefore, we have to analyse social processes. Accordingly, Wittgenstein's approach enables a purely interactionist view on processes of concept formation which are a benefit for the interpretative researcher, particularly as they are not dependent on speculations concerning student's thoughts. If the use gives meaning to words (in the interaction), then the (linguistic) action is the sole criterion for the reconstruction. Thus, we have to follow the ethnomethodological premise: The explication of meaning is the constitution of meaning. By analysing the students' "languaging" (Sfard 2008) for mathematical concepts, the development and alteration of meaning by the use of the according words, we are able to reconstruct the social learning in the mathematics classroom. Therefore, the qualitative interpretation of the classroom communication is founded on an ethnomethodological and interactionist point of view (cf. Voigt 1984; Meyer 2007). Symbolic interactionism and ethnomethodology build the theoretical framework which will be combined with the concepts of "language-game" and "(inferential) use".

The main aim of the presented study is to get a deeper insight into the processes of giving meaning to words in the mathematics classroom. Therefore, alternative ways of introducing concepts are going to be considered. Comparing possible and real language-games can help to understand the special characteristics of the actual played language-game.

The empirical data are taken from several studies in which the *arithmetic mean* was introduced. The surveys were carried out in two classes (first class: fourth grade in primary school, age of students: 9 to 10 years; second class: fifth grade, secondary school age of students: 10 to 11 years) on the one hand and in interviews with two students of the third grade in primary school (age: 8 to 9 years, duration: 3 x 45 minutes) each on the other hand. It was the empirical studies' aim to get the students to collect different judgments on one concept and give reasons for their relationship (in this situation: their equivalence). This means that in relation to the arithmetic mean, there are two judgments: Firstly, the arithmetic mean is the quotient of the total and the

number of the given values and secondly, the arithmetic mean is determined by the inversely arranging of the values to get an *adjusted value*. This can be expressed formally and briefly as follows ($\bar{x}$ is the arithmetic mean of $a_1,\dots,a_n$):

$$(a_1+\dots+a_n)=\underbrace{\bar{x}+\dots+\bar{x}}_{n\ summands}$$

In every experimental setting, the series of tasks start with a reduction of the meaning of the arithmetic mean to the meaning of a "middle number" in a number series (the students were told that the second box contains the number of the first):

$$5+6+7=\boxed{}\qquad\boxed{}:3=$$

By solving tasks like this the students should discover that the result of the division will be the "number in the middle", which could also be gained by modifying the summands to be equal to each other: 5+6+7 = (5+1)+6+(7-1) = 6+6+6. To get the general concept of the arithmetic mean, the summands, the amount of summands and the distance between the summands were varied gradually.

USE OF WORDS FOR CONCEPT FORMATION – EMPIRICAL EXCERPTS

By carrying out the empirical studies, learners were asked, among other things, to solve the following task – two different working-outs were to give: "Lisa weighs 14kg. Paul weighs 23 kg. Sarah weighs 25 kg. Marc weighs 26 kg. What is the middle weight?" Jule wrote about this:

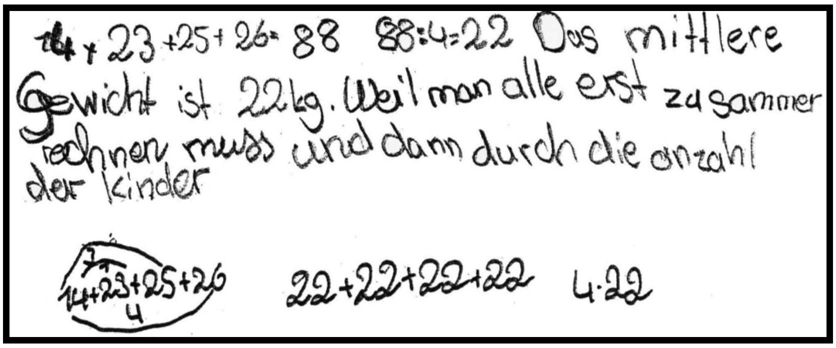

Figure 2: Jule (fourth grade) determines the arithmetic mean on two ways
(Translation: "The middle weight is 22kg. As you first have to calculate everything together and to divide it afterwards by the number of kids")

As suggested by the task, Jule speaks of "middle weight". Concerning this and other tasks, the students named the concept "number in the middle", "average", "balanced number", etc. Regardless of the name of the concept, the use of these words has been quite the same.

As an example for the different reasons for the equivalence of the two judgments of the arithmetic mean (first: quotient, second: inversely arranging), Malte's statements are

given in the following. Malte explains why the inversely changing of 12+14+16 does not change the total (the transcript has been translated and linguistically smoothed).

Malte: Eh, with plus it is a team so to say. The result always stays the same – no matter what is changed. If one doesn't take away anything and doesn't add anything, either- but if one always swaps, swaps, swaps, the result will always stay the same. With minus, it is different.

Teacher: Could you explain to me the thing you said about the team – What do you mean? […]

Malte: This (*pointing left to right at the task 12+14+16*) is the team now. And if this one (*pointing at the summand 16*) is now so to say- or- these (*again pointing left to right at the task*) are <u>the students</u>. This one (*pointing at 12*) is missing two pens and then this one (*pointing at 16*) who has two pens too much gives- one pen to this one (*pointing at 12*) who is missing two pens.

Students: 2 pens (murmuring)

Malte's argument can be reconstructed as follows:

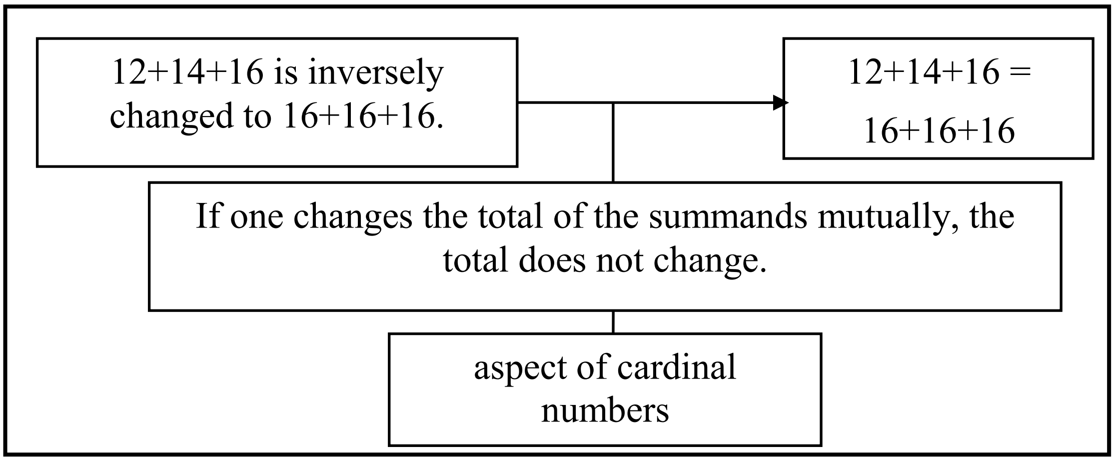

Figure 3: Reconstruction of Malte's argument

Malte uses words like *team* and *pen* to describe the remaining total of the inversely arranging. His given reasons can be put down to the aspect of the cardinal number in so far as he considers the change of the singular summands and not totals. In this way, he links both judgments of this pre-form of the *arithmetic mean*, which expands the concept's dimension. Later, Malte's argument (resp. its functional elements) is taken up by other students again and again (even there, where the students are able to distinguish between the arithmetic mean and the median) so that not only the connection, determined by Malte, is taken up constantly (cf. the above quotation by Brandom), but also the equivalence of both judgments is made clear by more elaborated arguments compared to those in Figure 3. Only the words *team* or *pens* as concrete objects of the change were not taken up. This can be interpreted as follows: The students refer to the way of the use and not only to the specific words which are used. In other words: The students seem to refer to Malte's implicit rule and his backing. Such moments are shown in the talk.

FINAL REMARKS

Wittgenstein's theory itself is not a theory of interpretation. Rather, he presents a theoretical framework which can be used on top of a theory of interpretation in order to understand processes of languaging for concept formation.

Corresponding to the given considerations based on the theories to Brandom, Toulmin and Wittgenstein concept formation can be understood as the (*inferential*) *use of judgments* and their (*general, regular*) *connections* (the *rules*). Throughout the arguments we commit ourselves to these judgments resp. to their connections which, if they are accepted, we can use continuously. This use can be, again, independent of the concrete words, but rather the more general way of using the words, the rules and backings, seem to be crucial for the (following) course of concept formation.

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